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J. Clemente-Gallardo, R. Lopezlena [1] , J.M.A. Scherpen
Delft University of Technology P.O. Box 5031, 2600 GA Delft, The Netherlands.
{J.Clemente-Gallardo, R.Lopezlena, J.M.A.Scherpen}@ITS.TUDelft.nl

Abstract

Based on the Stokes-Dirac structures proposed in [11, 10], an extension of these descriptions is presented for the Euler equations in the case of a one-dimensional manifold with boundary in order to include the entropy as a dynamic variable. Furthermore, a discretization procedure is proposed showing several nice properties. The effectiveness of the procedure for simulation purposes is shown with a simple model of an ideal pipeline.

1 Introduction

The aim of this paper is to introduce a geometry-based discretization procedure for the study of the Euler equations in a one dimensional manifold with boundary. The geometrical basis is the Hamiltonian description of Euler equations in terms of Stokes-Dirac structures proposed in [11, 10], slightly improved by the addition of the entropy as a dynamical variable.

The physical system to be studied is the motion of fluid in a one-dimensional manifold, with boundary, i.e. a manifold diffeomorphic to a closed interval of $\mathbb{R}$. This implies that we can consider the manifold to be Euclidean in what regards the metric structure.

In control theory, the discretization of infinite dimensional systems is a natural goal, in order to apply to their study the well-tested finite dimensional techniques. We are particularly interested in two of them: the control techniques based on passivity [7] and those of model reduction [4, 9]. Similar constructions have appeared recently applied to different systems [3], and though we use a similar original continuous model, we add a new ingredient to the description, and use a different discretization procedure. In our case, the discretization procedure is based on the substitution of the differentiable manifold M by a lattice of discrete points. As the continuous model uses the differentiable forms of M to model the physical magnitudes, and the discretization of the manifold yields a discretization of the exterior algebra, this is enough to discretize our model.

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The paper is organized as follows. In Section 2 we review some basic concepts about the Hamiltonian description of hydrodynamics, first in the Poisson framework and then in the Dirac framework. Later, in Section 3 we describe more carefully the new Stokes-Dirac structure that we propose for the description of a one-dimensional fluid and then, in section 4 we detail a general discretization procedure for the exterior algebra of a differentiable manifold. Finally, in section 5 we present the case of a one-dimensional fluid in detail, the experiment simulated on the computer and the corresponding results.

2 Hamiltonian description of hydrodynamics

2.1 The Poisson description

Throughout the paper we are going to consider the description of a fluid which moves in a para-compact manifold with boundary, or equivalently a Riemannian manifold. We also assume, for simplicity, that the manifold is orientable. Though so far we have only applied it to the one-dimensional case, the construction can be applied to any spatial dimension, and therefore we will keep the exposition at a general level as long as we can. We will use M to denote the n-dimensional manifold and ∂M to denote its boundary, whose dimension is $n-1$. Please notice that throughout the paper we use the term dimension to denote two different concepts: the n-dimensional manifold M represents the spatial domain where the motion takes place, but the system itself (the fluid) is infinite dimensional i.e. infinite degrees of freedom are required to model it. Our goal is to provide an accurate finite dimensional model for the fluid.

The description of the motion of a fluid is done in the literature (see [5, 1, 6]) in terms of the variables of the infinite dimensional Lie group $\mathrm{Diff}(M)$ defined by the diffeomorphisms of M. The situation is the infinite-dimensional analogue to what happens in the case of the (finite dimensional) rigid body, as it is explained in [5]. From the geometrical point of view, in both cases the Hamiltonian description is usually introduced in terms of the corresponding Lie-Poisson structure.

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2.2 The case with boundary: the Stokes-Dirac structure

In [11], the isentropic compressible fluid is studied under the framework of Stokes-Dirac structures. Roughly speaking, we can think on this type of descriptions as the Poisson structure together with the boundary conditions understood as constraints on the system. This point of view is very interesting for control purposes, since it allows us to consider interconnection of different systems or a control action on the boundary to obtain some desired response. We address the interested reader to [11] for the detailed exposition of the framework both for the geometrical and the physical points of view. We only like to point out that, in this framework, the physical magnitudes are modeled by means of the differentiable forms of the manifold M, and the structure is expressed in terms of geometrical objects. Our purpose now is to extend this description to include the entropy of the fluid.

3 Stokes-Dirac structure for fluid dynamics

In [11, 10] a compressible isentropic fluid is studied, and hence the two relevant magnitudes are the velocity and the density of each infinitesimal volume of the fluid. The density ρ is naturally modeled as a volume form in the spatial domain, while the velocity model is slightly more involved. The (Eulerian) velocity v of the system is naturally defined as a vector field on M, but since we need to express it as a differentiable form, we can use the Riemannian metric g that we assume the manifold to be endowed with, and define $v^\flat = g(v,\cdot) \in \bigwedge^1(M)$. [2] Now we extend the formalism introduced in [11, 10] to consider a fluid whose internal energy u exhibits a dependence on the entropy s. For the case where the boundary is non relevant there is a description in terms of Lie-Poisson structures (see [6, 1]). We now want to use Stokes-Dirac structures to model the system with free boundary conditions, since we want be able to, for instance, interconnect two such systems by joining their boundaries.

We continue with a focus on the one-dimensional spatial domain though it can be extended to higher dimensions. Consider as space of flows the following vector space:

$$\begin{aligned}\mathcal{F} &= \mathcal{F}_\rho \times \mathcal{F}_v \times \mathcal{F}_s \times \mathcal{F}_b \\ &= \bigwedge^1(M) \times \bigwedge^1(M) \times \bigwedge^0(M) \times \bigwedge^0(M)\end{aligned}$$

which represents, the velocity, the density, the entropy and the density in the boundary, respectively. The corresponding effort space will be written as

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_\rho \times \mathcal{E}_v \times \mathcal{E}_s \times \mathcal{E}_b \\ &= \bigwedge^0(M) \times \bigwedge^0(M) \times \bigwedge^1(M) \times \bigwedge^0(M).\end{aligned}$$

[2] Throughout the paper, we will omit the super-index $\flat$, unless it is confusing.

We can consider a product in the space $\mathcal{F} \times \mathcal{E}$ defined as:

$$\langle\langle (f^1,e^1),(f^2,e^2)\rangle\rangle = \int_M (f^1_\rho \wedge e^2_\rho + f^1_v \wedge e^2_v + f^1_s \wedge e^2_s) + \int_M (f^2_\rho \wedge e^1_\rho + f^2_v \wedge e^1_v + f^2_s \wedge e^1_s) + \int_{\partial M} (f^1_b \wedge e^2_b + f^2_b \wedge e^1_b) \tag{1}$$

where $f^i = (f^i_\rho, f^i_v, f^i_s, f^i_b)$ and $e^i = (e^i_\rho, e^i_v, e^i_s, e^i_b)$.

Definition 3.1 *A Dirac structure on $\mathcal{F} \times \mathcal{E}$ is defined as a maximally isotropic subspace of $\mathcal{F} \times \mathcal{E}$ with respect to the product (1).*

Our main geometric result is the following:

Theorem 3.1 *The subspace D satisfying*

$$\begin{bmatrix} f_\rho \\ f_v \\ f_s \end{bmatrix} = \begin{bmatrix} 0 & d & 0 \\ d & 0 & w \\ 0 & -w & 0 \end{bmatrix} \begin{bmatrix} e_\rho \\ e_v \\ e_s \end{bmatrix}$$
$$\begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_\rho|_{\partial M} \\ e_v|_{\partial M} \end{bmatrix}$$

where $w = (*ds)(*\rho)^{-1}$ and $*$ is the Hodge operator associated to the Riemannian structure which exists on M, defines a Dirac structure on $\mathcal{F} \times \mathcal{E}$.

Proof: The proof follows the lines of the one in [11] for the general case with two flows/efforts. Given a subspace $D \subset \mathcal{F} \times \mathcal{E}$, we define the space $D^\perp$ as:

$$D^\perp = \{(f,e) \in \mathcal{F} \times \mathcal{E} | \langle\langle (f,e),(f_D,e_D)\rangle\rangle = 0, \quad \forall (f_D,e_D) \in D\}.$$

Maximally isotropic subspaces are those for which $D^\perp = D$. This is the point to prove now, and we are going to do it in two steps: First, to prove that $D \subset D^\perp$. This is simple to state: given the points $(f,e) \in D$, we have to prove that the points of D annihilate them. Take then $(f^1,e^1) = (f^1_\rho, f^1_v, f^1_s, e^1_\rho, e^1_v, e^1_s, f^1_b, e^1_b) \in D$ and consider $(f^t,e^t) = (f^t_\rho, f^t_v, f^t_s, e^t_\rho, e^t_v, e^t_s, f^t_b, e^t_b) \in D$. The product $\langle\langle (f^1,e^1),(f^t,e^t)\rangle\rangle$ is

$$\int_M (f^1_\rho \wedge e^t_\rho + f^1_v \wedge e^t_v + f^1_s \wedge e^t_s) + \int_M (f^t_\rho \wedge e^1_\rho + f^t_v \wedge e^1_v + f^t_s \wedge e^1_s) + \int_{\partial M} (f^1_b \wedge e^t_b + f^t_b \wedge e^1_b).$$

Since the points belong to D, we can replace the expressions of the flows by the corresponding expressions in terms of the efforts. Then, we obtain:

$$\begin{aligned}&\int_M (de^1_v \wedge e^t_\rho + de^1_\rho \wedge e^t_v + *ds(*\rho)^{-1} e^1_s \wedge e^t_v \\ &\quad - *ds(*\rho)^{-1} e^1_v \wedge e^t_s) + \int_M (de^t_v \wedge e^1_\rho + de^t_\rho \wedge e^1_v \\ &\quad + *ds(*\rho)^{-1} e^t_s \wedge e^1_v - *ds(*\rho)^{-1} e^t_v \wedge e^1_s) \\ &\quad + \int_{\partial M} (e^1_v|_{\partial M} \wedge e^t_b + e^t_v|_{\partial M} \wedge e^1_b).\end{aligned}$$

Now, the only terms different from those in [11] are those containing the entropy. Then, it is trivial to see that the expression above reduces to:

$$\int_M (*ds(*\rho)^{-1}e_s^1 \wedge e_v^t - *ds(*\rho)^{-1}e_v^1 \wedge e_s^t) + \int_M (ds(*\rho)^{-1} * e_s^t \wedge e_v^1 - *ds(*\rho)^{-1}e_v^t \wedge e_s^1). \quad (2)$$

And since in this case, the efforts $e_v \in C^\infty(M)$ and $e_s \in \bigwedge^1(M)$ we have that $*ds(*\rho)^{-1}e_s^1 \wedge e_v^t = *ds(*\rho)^{-1}e_v^t \wedge e_s^1$ and $*ds(*\rho)^{-1}e_v^1 \wedge e_s^t = *ds(*\rho)^{-1}e_s^t \wedge e_v^1$. This implies that the contribution (2) is identically zero and hence proves the statement.

Now, we prove that $D^\perp \subset D$. We follow again the proof in [11] adapting it to our case. Consider a point $(f_\rho^t, f_v^t, f_s^t, e_\rho^t, e_v^t, e_s^t, f_b^t, e_b^t) \in D^\perp$ and a point $(f_\rho^1, f_v^1, f_s^1, e_\rho^1, e_v^1, e_s^1, f_b^1, e_b^1) \in D$. Our goal is to prove that $(f_\rho^t, f_v^t, f_s^t, e_\rho^t, e_v^t, e_s^t, f_b^t, e_b^t) \in D$.

Consider the product $\langle\langle(f^1, e^1), (f^t, e^t)\rangle\rangle$ given by

$$\int_M (f_\rho^1 \wedge e_\rho^t + f_v^1 \wedge e_v^t + f_s^1 \wedge e_s^t) + \int_M (f_\rho^t \wedge e_\rho^1 + f_v^t \wedge e_v^1 + f_s^t \wedge e_s^1) + \int_{\partial M} (f_b^1 \wedge e_b^t + f_b^t \wedge e_b^1) = 0.$$

Now, we can replace the flows (f_ρ^1, f_v^1, f_s^1) by their corresponding expressions in terms of the efforts, but we can not replace those of the second point. We obtain:

$$\int_M (de_v^1 \wedge e_\rho^t + de_\rho^1 \wedge e_v^t + *ds(*\rho)^{-1}e_s^1 \wedge e_v^t - *ds(*\rho)^{-1}e_v^1 \wedge e_s^t) + \int_M (f_\rho^t \wedge e_\rho^1 + f_v^t \wedge e_v^1 + f_s^t \wedge e_s^1) + \int_{\partial M} (e_v^1|_{\partial M} \wedge e_b^t + f_b^t \wedge e_b^1).$$

As the elements of each factor commute (they are a one form and a function), we obtain by comparing the coefficients of the same monomials in the efforts with super index "1":

$$f_\rho^t = de_v^t \quad f_v^t = de_\rho^t + *ds(*\rho)^{-1}e_s^t$$
$$f_s^t = -*ds(*\rho)^{-1}e_v^t \quad f_b^t = e_\rho^t|_{\partial M} \quad e_b^t = e_v^t|_{\partial M}$$

This concludes the proof. ∎

Remark: It is important to notice that in this case the Hodge operator is trivial due to the dimension of M.

The choice of the Dirac structure provides a Hamiltonian description of the dynamics (see [11]) for a Hamiltonian H defined as

$$H(\rho, v, s) = \int_M \left(\frac{1}{2}\rho g(v, v) + \rho u(*\rho, s)\right) \quad (3)$$

where g is the Riemannian metric, and u is the thermodynamical internal energy of the system, expressed in terms of the density and the entropy. The energy balance is given by:

$$\frac{dH}{dt} = \int_{\partial M} f_b \wedge e_b. \quad (4)$$

This expression coincides with the balance equation of the system without entropy, since our model implicitly assumes that the interchange of heat at the boundary is zero. This implies that the gradient of temperatures of the fluid at the boundaries vanishes, what constitutes a quite strong assumption that we hope to relax in the future. In any case, it is still reasonable to model the motion of gases in pipes controlled by valves.

4 Discretization

As we mentioned in the introduction, our discretization method is based in the discretization of the manifold M (as any finite element method). The important aspect of this choice is that the discretization of the exterior algebra of the system (i.e. the set of differential forms of M of any degree) is very simple and preserves some nice properties such as given by Stokes theorem.

4.1 Discretization of differentiable forms

Let us consider a lattice of points and links to represent our system. We assume that the spatial domain of the forms that represent our system is discretized and becomes a grid of points in n dimensions (where n is the dimension of the domain M). We also consider the links between the closest neighbors in the lattice, i.e. those connecting the points which are at a distance a of one given point.

We will denote the lattice of link length a as $\mathcal{L}_a^0$, and its points as $\sigma_{i_1 \cdots i_n}$. It is important to remember that we recover the continuum limit by taking a lattice with a distance $a = 0$.

In order to be consistent with this structure, we must provide a construction for the geometric objects of the exterior algebra of differential forms on this discretized framework:

- A function on our system is defined as a set of real values defined on the nodes of the lattice, i.e. as a mapping $f : \mathcal{L}_a^0 \to \mathbb{R}$.
- Covectors at one point are defined by using a natural basis: we consider the basis vectors to be defined on the middle point of the links, and hence the covectors will be just real numbers defined on them. The reason for the choice of the middle points will be clear when we define the exterior derivative below. The middle points of the links

define a new lattice that we denote as $\mathcal{L}_a^1$ (the points of this new lattice are the middle points of the links of $\mathcal{L}_a^0$). The one form is hence defined as the mapping $\alpha : \mathcal{L}_a^1 \to \mathbb{R}$.

- The exterior derivative is defined according to the basis chosen above. Given a function f on a manifold M, its exterior derivative is the one form defined as: $df = \frac{\partial f}{\partial x^i} dx^i$ in a local basis of the cotangent bundle T^*M. If we want our lattice to be the discretized version of M, the elements of the basis dx^i become the elements defined on the middle points of the links of $\mathcal{L}_a^0$. Now we have to discretize the partial derivative. One of the possible choices (there are many) is to consider:

$$\frac{\partial f}{\partial x^i} \to \frac{f_{k_1 \cdots k_{i+1} \cdots k_n} - f_{k_1 \cdots k_i \cdots k_n}}{a}$$

  This definition also implies a choice of an orientation for the links. Finally, we can conclude writing the exterior derivative acting on functions $d : \mathcal{L}_a^0 \to \mathcal{L}_a^1$ as $df = \frac{f_{k_1 \cdots k_{i+1} \cdots k_n} - f_{k_1 \cdots k_i \cdots k_n}}{a} dx^i$, i.e. a number $(\frac{f_{k_1 \cdots k_{i+1} \cdots k_n} - f_{k_1 \cdots k_i \cdots k_n}}{a})$ that multiplies the element of the basis dx^i.

- Higher degree forms are defined in an analogue way, always according to the choice of basis we did before: we define the basis to be some specific points where we place the elements of the basis. The forms are specified by their coordinates (real numbers) on all these points. Two forms are defined on the center of the squares of lattice $\mathcal{L}_a^0$, and with an orientation (clockwise or anticlockwise). The center of the squares defines a new lattice, $\mathcal{L}_a^2$, and in its vertex the real numbers which correspond to the two form $\beta : \mathcal{L}_a^2 \to \mathbb{R}$ are defined. Of course, the exterior derivative must connect one forms and two forms, and this relation is as follows: consider a one form $\alpha : \mathcal{L}_a^1 \to \mathbb{R}$ as real values defined at the nodes of the lattice $\mathcal{L}_a^1$. Consider the exterior differential to act on them exactly as it does when acting on a function to define one forms, i.e., given a one form $\alpha = \alpha_s dx^s$ and $d : \mathcal{L}_a^1 \to \mathcal{L}_a^2$ we have $df = \frac{\alpha_{k_1 \cdots k_{i+1} \cdots k_n} - \alpha_{k_1 \cdots k_i \cdots k_n}}{a} dx^i \wedge dx^s$; where the indices refer to the $\mathcal{L}_a^1$ lattice.

- The rest of the exterior algebra is defined analogously. Three forms are defined on the center of the cubes of side a (this defines a new lattice $\mathcal{L}_a^3$), four forms on the center of the hypercubes (lattice $\mathcal{L}_a^4$), etc.

4.2 The case of a one-dimensional lattice: the Dirac structure

Since we are interested in a one-dimensional system, let us consider now a one-dimensional lattice where only functions and one-forms are defined. Functions are mappings $f : \mathcal{L}_a^0 \to \mathbb{R}$ while one-forms are defined as $\alpha : \mathcal{L}_a^1 \to \mathbb{R}$. Besides, the exterior derivative of functions defines one-forms, as we saw above: $d : \mathcal{L}_a^0 \to \mathcal{L}_a^1$. Suppose now that in this framework we want to find the discretized expression corresponding to the Stokes-Dirac structure in (2). Let us assume that we discretize the manifold M and we obtain a linear lattice of link length a. We denote such a lattice as $\mathcal{L}_a^0$ as we did above. The important point is that, once we have chosen the lattice, there is a natural choice for the set of functions (real numbers on the sites of the lattice $\mathcal{L}_a^0$) and for the one-forms (real numbers on the lattice $\mathcal{L}_a^1$).

As an example, consider a discretization with only two points (the boundary points). In such a case, our lattices are: $\mathcal{L}_a^0 = \{\{p_1\}, \{p_2\}\}$ and $\mathcal{L}^1 = \{(\{p_1\} + \{p_2\})/2\}$. The corresponding vector spaces are just the set of real functions on those points. We must consider two copies of the lattice and the two points isolated to represent the vector spaces of flows and efforts (which are the set of functions and one-forms defined as we saw above): $\mathcal{E} = \mathcal{L}_a^0 \times \mathcal{L}_a^0 \times \mathcal{L}_a^1 \times \mathcal{L}_a^0$ and $\mathcal{F} = \mathcal{L}_a^1 \times \mathcal{L}_a^1 \times \mathcal{L}_a^0 \times \mathcal{L}_a^0$.

The expression of the Dirac structure D then becomes,

$$\begin{bmatrix} -\frac{1}{a} & \frac{1}{a} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{a} & \frac{1}{a} & \kappa & 0 & 0 \\ 0 & 0 & 0 & 0 & -\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

with $\kappa = \rho^{-1} a^{-1} (s_1 - s_2)$ and the corresponding relation between flows and efforts $f = Pe$, with $f = (f^\rho, f^v, f_1^s, f_2^s, f_1^b, f_2^b)^T$, $e = (e_1^\rho, e_2^\rho, e_1^v, e_2^v, e^s, e_1^b, e_2^b)^T$ P given by

$$\begin{bmatrix} 0 & 0 & -\frac{1}{a} & \frac{1}{a} & 0 & 0 & 0 \\ -\frac{1}{a} & \frac{1}{a} & 0 & 0 & \kappa & 0 & 0 \\ 0 & 0 & 0 & -\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & -\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

where $\{e_1^\rho, e_2^\rho\} \in \mathcal{L}_a^0$ (the first copy of the lattice functions), $\{e_1^v, e_2^v\} \in \mathcal{L}_a^0$ (the second copy of the lattice functions), $e_s \in \mathcal{L}_a^1$, $\{e_1^b, e_2^b\} \in \mathcal{L}_a^0$ (the copy of the lattice functions corresponding to the boundary), $f_1 \in \mathcal{L}_a^1$ (the first copy of the one forms), $f_2 \in \mathcal{L}_a^1$ (the second copy of the one forms) and $\{f_1^b, f_2^b\} \in \mathcal{L}_a^0$ (the flows at the boundary).

It is straightforward to see that the interconnection of two such objects (in the sense of [11], i.e. asking the boundary variables of both systems to satisfy certain relations), provides a new discretization with a lattice of three points. Hence, we can consider a discretization

Table 1: Values of physical variables

Variable		Value	Units
Density	ρ	1.2929	Kg/m^3
Moles	m	28.97	Kg
Velocity	v	10	m/s
Pressure	p	109	kPa
Sp. entropy	s	50	J/Kg
Thermal conduct.	k	1×10^8	W/mK
Temperature	T	293.15	K
Total length	ℓ	50	m

of any number of points to be again an interconnection of many systems as the one above. Of course, it follows that when the number of points goes to infinity (or equivalently, the length $a \to 0$) we obtain the original infinite-dimensional system.

5 Simulation results

In this section an implementation of the discretization procedure is presented in order to simulate the behaviour of a one-dimensional system approximated by n-points. Discretization of the efforts $(\delta_\rho H, \delta_v H, \delta_s H)$ resulting from the Hamiltonian yield for $i = 1, 2, \cdots, n$,

$$\delta_\rho H_i = \frac{1}{2}v_i^2 + \frac{1}{\rho_i}P_i + u(\rho_i, s_i), \quad (5)$$

$$\delta_v H_i = \rho_i v_i, \quad (6)$$

$$\delta_s H_i = \rho_i T_i. \quad (7)$$

where T_i is the temperature of the system obtained as the derivative $T = \frac{\partial u}{\partial s}$, pointwisely, and u as in (3). The equations of motion turn out to be

$$\frac{\partial \rho_i}{\partial t} = \frac{-1}{a}\left(\delta_v H_i - \delta_v H_{i-1}\right),$$

$$\frac{\partial v_i}{\partial t} = \frac{-1}{a}\left[\delta_\rho H_i - \delta_\rho H_{i-1} + \delta_s H_i \frac{s_i - s_{i-1}}{\rho_i}\right],$$

$$\frac{\partial s_i}{\partial t} = \frac{-1}{a}\left[\delta_s H_i \frac{s_i - s_{i-1}}{\rho_i}\right].$$

with the temperature determined by the Hamiltonian (3) and the usual thermodynamic relations as [2]:

$$\frac{\partial T_i}{\partial t} = \frac{k}{\rho_i C_p}\frac{T_{i+1} + T_{i-1} - 2T_i}{a^2} - v_i \frac{T_i - T_{i-1}}{a}.$$

The boundary conditions are determined at $i = 0$ by fixing in eqs. (5)-(7) the values $\rho(0) = \rho_0$, $v(0) = v_0$, $P(0) = P_0$, $T(0) = T_0$ and $s(0) = s_0$. At $i = n$, $T(n) = T_n$.

Consider the description of the model for a 50 m of length ideal pipeline transporting atmospheric (dry) air at standard conditions (see Table 1). The pressure is determined with the ideal thermodynamic equation of state $p = \rho RT/M$. Nevertheless any other equation of state could be used (e.g. Soave-Redlich-Kwong or Peng-Robinson) with increased numerical precision on the determination of pressure [8]. This modelling approach may be attractive in several applications where, under the restrictions previously stated, interconnection of fluid systems is needed and the Hamiltonian structure becomes useful, for instance, gas distribution networks, waterhammer studies, or the study of the rarefaction wave propagation caused by a leak. The previous equations were implemented in the Matlab/Simulink programming platform with the integration routine Runge-Kutta 45 for $n = 100$, $a = 0.5\,m$ determining for each node the pressure using the thermodynamic equation of state. The simulation assumes that initially the fluid is flowing at a speed of $10 m/s$ at steady state. Therefore every transient has a delay of approximately 5 s at the end of the pipeline. At $t = 0s$ a step variation in the boundary mass flow of 0.2 s of duration increases the speed 50% of the nominal flow (see fig. 1). As a result a transient impulse of density was propagated along the pipeline (see fig. 2), with a small transient in temperature, fig. 3.

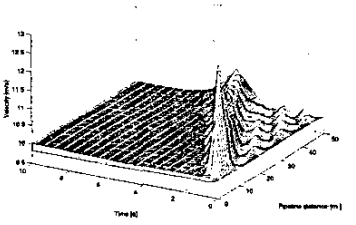

Figure 1: Velocity in simulation example

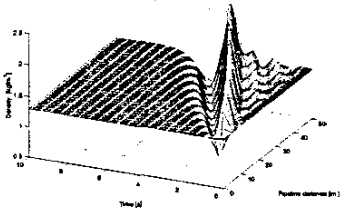

Figure 2: Density in simulation example

In order to show the effect on the variation of entropy, an initial condition of specific entropy with sinusoidal variations of 0.1% along the pipeline, was posed. A wavy transient can be seen in all graphics due to the transient effect of non uniform entropy, which after the first 5 s of simulation is reduced to a constant value, arriving to an homentropic condition.

Since port-Hamiltonian fluids preserve energy, the increase of input velocity reflects as a transient increase

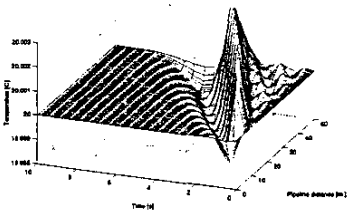

Figure 3: Temperature in simulation example

of energy as can be seen in fig. 4, due to transient responses in density and temperature (see figs. 2 and 3) which after propagating along the pipeline (in approximately 4.8 s) newly returns to a condition of physical constant energy. A closer inspection of fig. 4 shows that during the first 0.5s the step produces a ramp that steadily increases the level of energy and this value is approximately preserved during the time it takes this transient to travel along the pipeline. The wavy variation at the highest value of energy is due to the transient associated to the variation of entropy. Notice that at the end of this transient (at $t = 5\ s$) a transient decrease of energy is experienced due to the compressibility of the fluid. This effect can also be seen in figs. 2 and 3. At approximately 8 s the system returns to a constant value, which is the original steady-state energy.

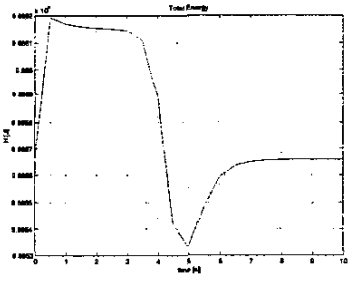

Figure 4: Total energy in simulation example

6 Future research

The discretization method of infinite-dimensional port-Hamiltonian systems presented in this paper is applicable to more general systems than given in our simulation example. We have chosen the example of the ideal gas for simplicity, but more realistic examples are implementable and more interesting from the practical point of view. An additional issue that needs further study is related to a condition for the variables at the boundary. The models presented here have variables at the boundary that just take care of the interchange of kinetic energy. The model requires that the fluid, which enters the system through the boundary, has the same temperature as the system inside, or in other words, we assume a completely adiabatic filter closing the boundary. Relaxing this condition is a topic of ongoing research.

Other generalizations that are in progress are: firstly, the extension to higher dimensions in the simulation example. The 2D-case is being adapted right now, and the 3D-case will be our next goal. Secondly, addition of energy dissipation is necessary. Real applications require energy dissipation in the form of heat caused by the friction of the fluid with the pipe walls and the inclusion of viscous terms in the momentum equation. These problems are part of the ongoing research topics. Finally, we pretend to further reduce the order of states without loosing the physical structure, but also taking the input-output behaviour into account. The balancing procedure given in [9], [4] needs to be further modified for the fluid-type of systems under consideration.

References

[1] V. I. Arnold and B.A. Khesin, *Topological methods in hydrodynamics.* Springer, 1998.

[2] J.H.Ferziger, M. Peric, *Computational Methods for Fluid Dynamics*, 2nd ed, Springer, Germany, 1999.

[3] G. Golo, V. Talasila, A.J. van der Schaft, B.M. Maschke, "Hamiltonian discretization of the Telegrapher's equation," *submitted*, 2002.

[4] R. Lopezlena and J. M.A. Scherpen, "Energy functions and balancing for nonlinear discrete-time systems," *IFAC World Congress*, Barcelona, 2002.

[5] J. E. Marsden, T. Ratiu, and A. Weinstein, "Semidirect products and reduction in mechanics," *Trans. Amer. Math. Soc.*, 281:147–177, 1984.

[6] S.P. Novikov, "The Hamiltonian formalism and a many-valued analogue of Morse theory," *Russ. Math. Surveys*, 37(5):1–56, 1982.

[7] R. Ortega, A.J. van der Schaft, B. Maschke, and G. Escobar, "Interconnection and damping assignment passivity-based control of port controlled Hamiltonian systems," *Automatica*, 38(4):585–596, 2002.

[8] R.C. Reid, J.M. Praushnitz, T.K. Sherwood, *The properties of gases & liquids*, 4th. Ed., McGraw-Hill, Singapore, 1987.

[9] J. M.A. Scherpen. "Balancing for nonlinear systems," *Syst. Contr. Letters*, 21:143–153, 1993.

[10] A.J. van der Schaft and B. Maschke, "Fluid dynamical systems as Hamiltonian boundary control systems," *Proceedings of CDC 2001*, Orlando Fl., p4497-4502, 2001.

[11] A.J. van der Schaft and B. Maschke, "Hamiltonian formulation of distributed-parameter system with boundary energy flow," *J. Geom. Phys*, 42, p166-194, 2002.